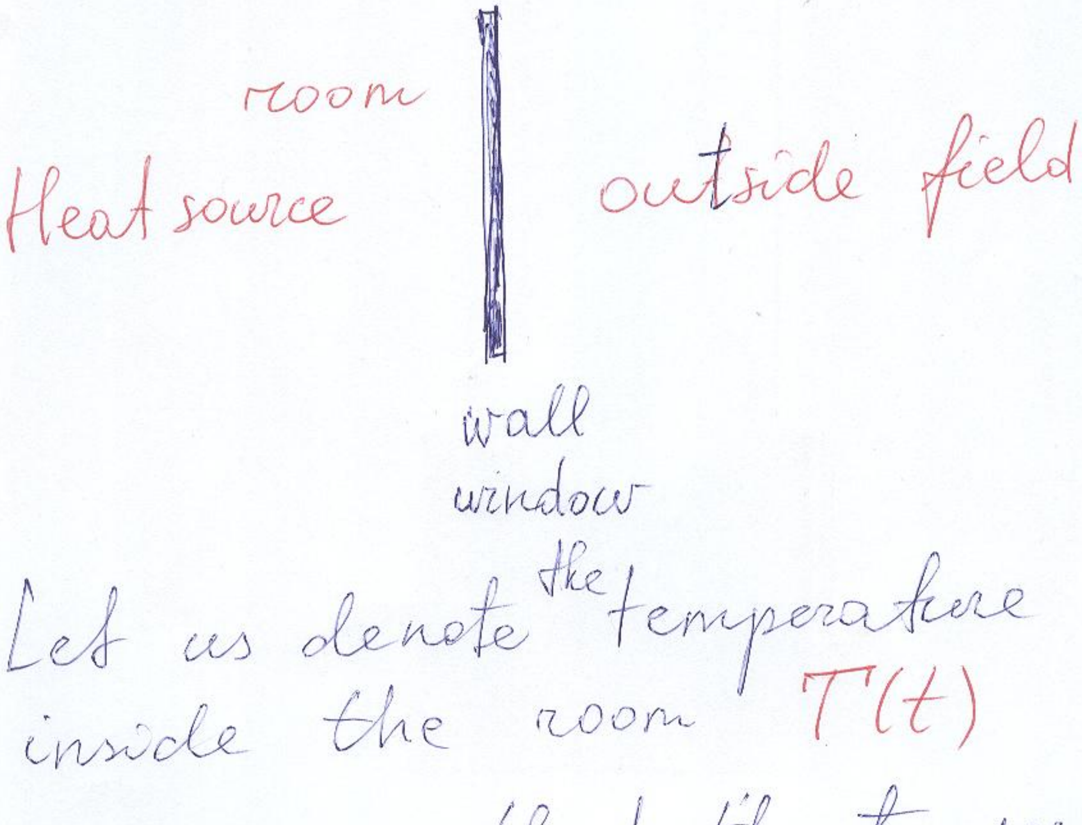
Lecture 3

Examples of ODE type models Formulation and Analysis

Model 1 Heat conduction



Let us denote the temperature inside the room $T(t)$

We assume that the temperature outside is constant T_L.

The heat soure $f(t)$.

Physics of this process:

a) the amount of heat conducted from a room outside can be calculated using Fourier's Law of conduction, which determines the rate heat transfer

$$Q(t) = r\left(T(t) - T_L\right)$$

where r defines the thermal conductivity.

b) the total amount of heat changes due the heat produced by a heat generator and due ~~to~~ the flow of ~~to~~ heat ~~to~~ outside from a room.

c) initial temperature $T(0) = T_0$.

An energy balance during a small time interval Δt can be expressed as

$$\begin{pmatrix}\text{Rate of change} \\ \text{of the energy} \\ \text{content}\end{pmatrix} = \begin{pmatrix}\text{Rate of heat} \\ \text{generation}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat} \\ \text{conduction}\end{pmatrix}$$

The rate of change of the energy

$$\rho c_v \frac{dT}{dt},$$

where

ρ is a density of material/air.
$\left[\frac{kg}{m^3}\right]$

c_v - is the specific heat
$[J/(kg\,K)]$.

Let denote q the density of heat generation

$$[W/m^3]$$

Thermal conductivity $\varkappa$ is measured in (density):

$$[W/(m^3 K)]$$

From the balance equation we get the mathematical model:

$$\begin{cases} \rho c_v \dfrac{dT}{dt} = q - \varkappa (T - T_i), \\ T(0) = T_0 \end{cases}$$

First we should check if the dimensions of each term are the same

$$\left[\rho c_v \frac{dT}{dt} \right] = \frac{kg}{m^3} \times \frac{J}{kg \cdot K} \times \frac{K}{s}$$

$$= \frac{J}{m^3 \cdot s}$$

$$\left[\kappa T \right] = \frac{W}{m^3 \cdot K} \times K = \frac{W}{m^3} = \frac{J}{m^3 \cdot s}.$$

$$\left[q \right] = \frac{W}{m^3} = \frac{J}{m^3 s}.$$

The balance of energy includes terms of the same dimension

Mathematical Model of parachute jumper

A mathematical model of parachutes uses Newton's Second Law.

We consider balance equation of downward gravitational force

$$F = mg$$

with upward aerodynamic forces (drag and lift)

$$F = -\beta v^2$$

Model

$$\begin{cases} m\dfrac{dv}{dt} = mg - \beta v^2 & V(0) = V_0 \\ v = \dfrac{dx}{dt} \\ & x(0) = X_0. \end{cases}$$

Let us consider the vectorial field

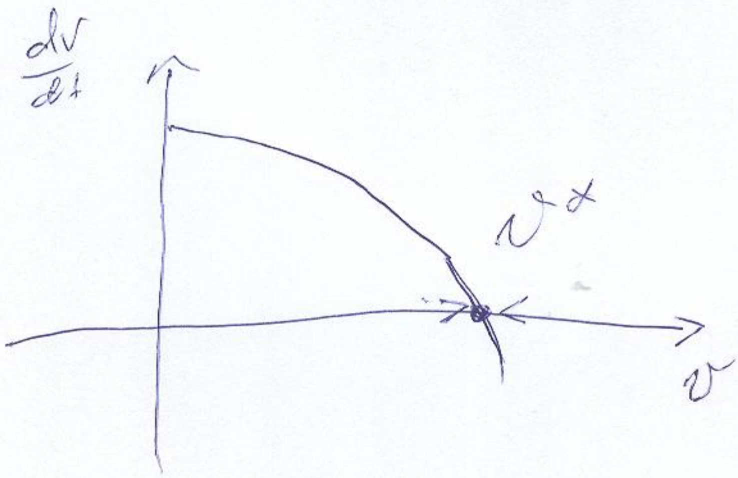
$$f(v) := \frac{dv}{dt}$$

$$\Rightarrow f(v) = g - \frac{\beta}{m} v^2$$

The fixed (stationary) point is obtained by solving equation

$$f(v^*) = 0$$

$$\Rightarrow v^* = \sqrt{\frac{mg}{\beta}}$$



v^* is a stable fixed point

Let denote

$h(t)$ - the distance from the Earth surface.

Mathematical model:

$$\frac{dh}{dt} = -v, \quad h(0) = H.$$

Questions:

1) How long will be the jump process
2) Find $v(T)$ at the moment you will reach the Earth:

$$h(T) = 0$$

3) Check if v^* (stability value) is reached and when?